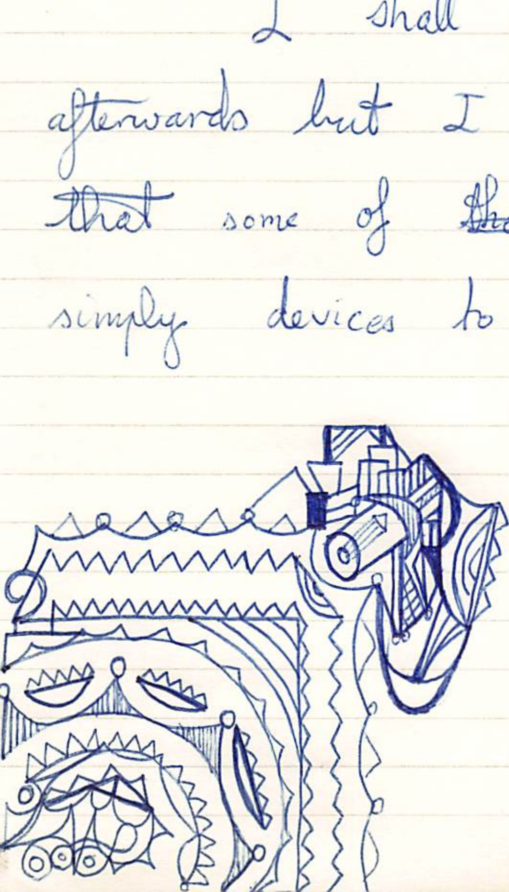
Moral Foundations of Majority Rule

In this paper I shall prove some propositions about the properties of ~~a~~ majority voting as a method of taking collective decisions compared with some other possible methods. Unless I have made any slips in arithmetic everything I say will be definitely true, in the sense ~~that~~ it will follow logically from the premises. ~~Whether~~ How far what I shall be saying ~~d~~ has relevance to the real world depends on the ~~acceptability~~ realism of the premises, and this is limited; how far the model itself is useful depends on whether ~~unrealistic~~ it is possible to introduce more realistic premises ~~a~~ while maintaining a manageable degree of complication. This can be done easily with some, ~~with~~ but ~~not others~~ ~~are more difficult~~ not all.

Let me be specific. ~~In~~ I shall be assuming ~~that~~ we have a population ~~with~~ each member ~~each~~ of which has definite views on a number of issues, that on any given issue ~~can~~ all the views can be expressed on a single dimension, and that on any issue the positions taken up by the members ~~the population falls into not more than five groups on~~ of the population fall into not more than five different positions on the ~~spectrum~~ single dimension. I shall further assume ~~that~~

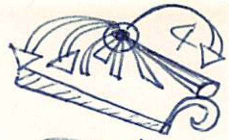
each member of the population gets the same amount of satisfaction ~~when~~ from a collective decision which coincides with his own first preference and ~~that~~ satisfaction from other outcomes falls off for everyone according to ~~on~~ one of two linear functions. ~~I shall~~ So much for the premises. There are two further limitations on the analysis. First, I shall consider only ways of aggregating preferences which which are one-stage processes e.g. voting directly on issues. I ~~thus~~ exclude all multi-stage processes such as voting for representatives who then take collective decisions. ~~by some means~~. And second, ~~Finally~~ in evaluating alternative methods of aggregating preferences I shall assume ~~that~~ the only relevant criterion is the amount and distribution of satisfaction among the population.

I shall be happy to discuss these limitations afterwards but I think it is just worth saying now ~~that~~ some of ~~them~~ the most severe limitations are simply devices to help the exposition. Thus, the ~~only~~ number of possible positions can be increased from five to infinity without any substantial change in the analysis itself, and the number of linear functions for loss of satisfaction



The idea that everyone derives the same satisfaction from his most preferred outcome is not necessary at all and simply helps the exposition — we could formulate the whole thing in terms of loss of satisfaction compared to the most preferred outcome.

can also be increased indefinitely. (Non-linear functions are also possible though some of the results then change.) The restriction of issues to one dimension is less trivial to get rid of, but with some further specifications more dimensions can be handled. What the model can't handle is issues which do not allow the population to be arrayed on dimensions at all. Without too much complication we can allow for the fact that some people have no preferences on some issues by saying they derive the same satisfaction from any outcome. What we cannot allow for ~~without~~ within the model is that people's preferences may be so ill-informed that the satisfaction they get from their preferred outcome is illusory or their indifference between outcomes is mistaken. This of course reflects on the plausibility of the evaluative ~~as~~ criteria. ~~too~~ All we can do here is point out that the model has less applicability in a given situation to the extent that the population are ill-informed about the implications of alternative outcomes. Finally, the limitation to one-stage processes of collective decision can be abandoned provided a specific theory is produced about the way the ~~two-stage~~ multi-stage procedure is expected to work, but ~~it is perhaps worth saying that~~ the whole



business becomes so much more complicated that it is best to regard it as an entirely new set of models which have some relationships to the single-stage one. ~~However, there are~~ ~~However, some of the points I shall be making~~ ~~ever~~ However, on certain assumptions which are not too far-fetched we can say that the results of voting for representatives who then themselves take collective decisions by voting is ~~not so~~ broadly similar to the results of voting directly on issues by the population itself. To the extent that this is so one can use the analysis here of majority voting on issues to get some idea of the advantages and disadvantages of representative democracy too. Summing up, the model to be developed here fits most closely decision-making in a small committee of well-informed people. What relevance if any it has for ~~representative democracy~~ ~~with~~ mass electorates is something we might discuss later. It ~~was~~ is worth pointing out though that a good deal of normative work on democracy makes assumptions which are not much less severe than mine, though they are usually made less explicit.

If this goes out of the window, therefore, a whole lot of utilitarian or interest-based ~~work~~ justifications of democracy go out of the window too. Perhaps it should. But I think one good way of finding out is to make the underlying assumptions explicit and see what we get.

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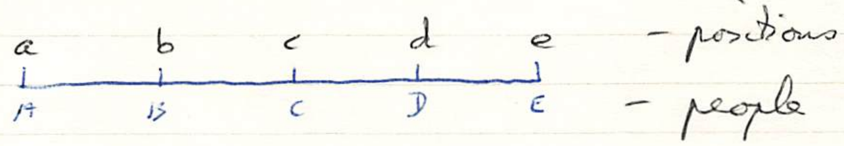
I shall introduce the criteria for evaluating procedures by applying them to the simplest kind of case. Suppose we have five people (or five equally-sized groups of people) distributed at equal distances along a single dimension in relation to some issue.

Fig 1



And let's say that they have an equal intensity of feeling about the issue. Following the treatment of intensity already outlined this means that the satisfaction they derive from the outcome declines at the same linear rate with the distance their own most preferred position is from the outcome. ~~And~~ Let's call the satisfaction each derives from his most preferred position 8 units

~~units~~ of satisfaction (this is quite arbitrary but has a useful property in our examples) and say that this satisfaction declines by one unit ~~for each~~ if the outcome is at ~~X~~ an adjacent position, by two units if the outcome is two places away, and so on. Thus, B gets 8 ~~10~~ units if the outcome is at b~~B~~, ~~9~~7 units if it is at a~~A~~ or c~~C~~, 6 ~~8~~ units if it is at ~~D~~d and ~~7~~5 units if it is at ~~E~~ e.

~~The~~ Now take the simplest decision-procedure: simple majority voting. It is known that if all the possible outcomes are paired with one another one will beat all the others and that is the position of the median voter, in this case C. If the procedure followed does not pair all the alternatives against one another it is possible for the result to be something else – for example if the vote is simply between b~~B~~ and ~~E~~e then b~~B~~ will get a majority. But it is a reasonable expectation that the ~~po~~ outcome will be at ~~C~~b most of the time. Given an outcome at ~~C~~b we can compute the satisfaction scores for each of the participants as follows:

A	~~8~~	6
B	~~9~~	7
C	~~10~~	8
D	~~9~~	7
E	~~8~~	6

Let us now apply our criteria. These are as follows:

(1) <u>Efficiency.</u> By efficiency is meant here the average score of the participants. In this case it is 6·8 units.

(2) <u>Minimum.</u> The second criterion is the score of the worst-off person. Here it is 6 [illegible] units. The significance of the situation of the worst-off person is ~~that~~ one may wish to reject a procedure even if it is quite efficient if it makes some people very badly off.

(3) <u>Equality.</u> By equality I mean here the average deviation in units from the ~~average~~ mean score. ~~divided by the ~~average~~ mean score.~~ Here the average deviation from the mean score is ~~[illegible]~~ ·64.
~~which divided by 6·8 is ~~·073~~ ·09~~

Now consider some alternatives to simple majority voting. These need not be actual operational procedures – it is ~~still interesting to ask how majority voting compares with~~ sometimes suggested ~~that~~ although majority voting may be the best operational procedure we could do better with ~~an all~~ a benevolent and perfectly informed dictator applying some calculus to people's preferences and it is interesting to see if this is true.

(1) Maximize efficiency. What outcome would provide the ~~most effi~~ highest average score? The answer is in fact c, that is, the same as majority voting produces. The average with c is 6·8 units, as I've mentioned; with b or d it's 6·6 and with a or e ~~it's~~ it drops off much faster to 6·0.

(2) Maximize the minimum. Here again among single outcomes c, the majority voting outcome, ~~comes~~ is top. With c the lowest score ~~(achieved by A and E)~~ is 6; with b or d it's 5, and with a or e it's 4. If we take randomized mixtures outcomes there are various ways of getting a minimum expected value of 6 but no way of getting a higher one.

(3) Maximize equality. Here the best outcome is not c but a coin-flip between a and e.

		outcome		
		a	e	$\frac{1}{2}$ a + $\frac{1}{2}$ e
person	A	8	4	6
	B	7	5	6
	C	6	6	6
	D	5	7	6
	E	4	8	6

This, as we see, produces an expected value of exactly 6 for each person. Notice, though, that this achievement of exact equality is brought about simply by destroying satisfaction on the part of those in the inner positions on the spectrum. A and E get the 6 units which they would get anyway ~~in~~ with the most efficient outcome, c, and all ~~that~~ happens is ~~that~~ B and D lose ~~two~~ one unit and C two units of satisfaction. ~~This in fact~~ Thus equality does not improve the worst off but merely damages the better off; this is in fact implied by our finding ~~that~~ there is no way of giving the worst-off person more than 6 units, which the worst-off person gets anyway ~~in~~ with the most efficient outcome in the position. This seems to me a fairly striking and counter-intuitive result, which greatly strengthens the claims of majority voting in situations with an even dispersal of opinion and a uniform intensity of feeling about the issue. How it fares with uneven distribution of opinion and/or unequal intensities of feeling is of course the nub of the matter. I shall be able to show ~~that~~ the superiority of majority voting is pretty insensitive to uneven distributions of sentiment but rather more affected by unequal intensities.

(4) <u>Equalize the chances of winning.</u> This is an ingenious idea which as far as I know was invented by James Coleman. The idea here is ~~that~~, especially if sentiments are distributed the same way over many issues, there is something unfair about the people who happen to be in the median position always winning, ~~that~~ is to say, always having the collective outcome coincide with their own preferred position. Wouldn't it therefore be more fair to have each outcome occur with a probability proportional to the number of people whose first preference it is? This involves collecting all the views and then operating some randomizing device which will produce an outcome supported by x per cent of the population x per cent of the time. We can easily work out the consequences of adopting such a procedure for our present case: since there are five people (or five equally sized groups of people) we can expect in the long run each possible outcome a, b, c, d and e will come up one fifth of the time.

	outcome					
	a	b	c	d	e	$\frac{1}{5}a+\frac{1}{5}b+\frac{1}{5}c+\frac{1}{5}d+\frac{1}{5}e$
A	8 10	7 9	8 6	7 5	6 4	6 8·0
B	7 9	8 10	9 7	8 6	7 5	6 8·6
C	6 8	7 9	10 8	9 7	8 6	6 8·8
D	5 7	6 8	9 7	10 8	9 7	6 8·6
E	4 9	5 7	8 6	9 7	10 8	6 8·0
						6·4

The results of this procedure are quite interesting. Clearly equal winning doesn't produce equal satisfaction, but it produces more equal satisfaction than does majority voting: the average deviation from the mean score is ·32. ~~and the average deviation divided by the mean score is~~ What this shows is that equal winning is a sort of half way house between ~~majority voting~~ efficiency (ie the median winning) and equal satisfaction (ie the two extremes winning an equal amount of the time). This is confirmed when we take efficiency too.

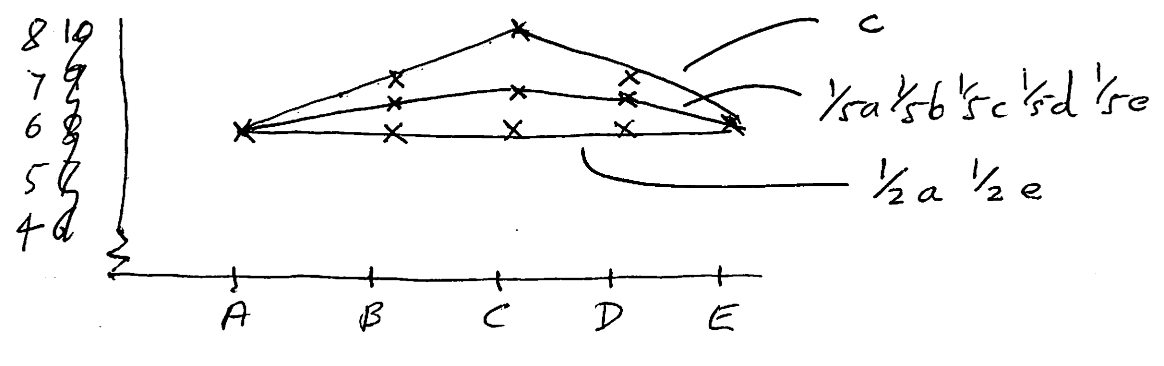
	Efficiency	Equal winning	Equal satisfaction
Average score	6·8	6·4	6·0
Av deviation	0·64	0·32	0·0
~~Av deviation / Av Score~~	~~0·9 [illegible]~~	~~0·5 [illegible]~~	~~0·0~~
Minimum score	6 8·0	6 8·0	6 8·0

Notice ~~that~~ the lowest score in each case is 68·0. When I discussed the criterion of maximizing the minimum I said ~~that~~ the best single position is c but ~~that~~ there were mixtures of positions capable of producing an equally good minimum. We can now see ~~that~~ both $\frac{1}{2}a\,\frac{1}{2}e$ and $\frac{1}{5}a\,\frac{1}{5}b\,\frac{1}{5}c\,\frac{1}{5}d\,\frac{1}{5}e$ produce exactly the same minimum of 81.6.

(5) <u>Take the average position.</u> The final criterion I shall consider is that of taking the average position. One might arrive at this as a prescription ~~either~~ out of sense of fairness — somehow the average might be thought of as ~~including more information abo~~ taking more account of the dispersion of people over positions than the median which is the result of majority voting. Specifically one might expect it to produce more equality & a higher minimum. Or one might arrive at it by a guess that ~~it~~ taking the average ~~would be~~ would tend to be more efficient than taking the median, which results from majority voting. I mention this fifth criterion here because I shall return to it later but it is I suppose obvious ~~that~~ with our present kind of ~~an uneven~~ dispersion of sentiments along

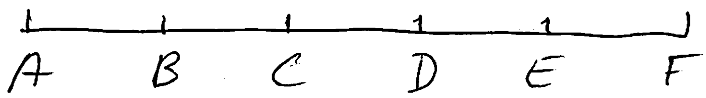
the continuum the average and the median coincide at c. In the present case, therefore, the two produce the same answers but whenever we have ~~a~~ some bunching of ~~[illegible] distribution~~ of sentiments ~~we~~ the median and average are likely to diverge. (I'll explain exactly how ~~to~~ I compute the average when we get to a relevant example.)

We can present the results so far in graphical terms, representing the people along the horizontal axis and their units of satisfaction under various methods of aggregating preferences on the vertical axis.



This shows clearly ~~that~~ you can have complete equality or maximum efficiency or something in between (equal winning) but they all leave A and E with 6 units of satisfaction and the only question is whether the others get more than ~~that~~ and if so how much.

I've worked this all out for five equally-distant and equally-populated positions but all the points I have made hold for any odd number of equally-distant and equally-populated positions except where there is just one position. ~~one~~ When we have an even number there is a slight complication in ~~that~~ there are two median positions. If we suppose ~~that~~ with majority voting ~~[illegible]~~ ties are broken by a random procedure, or merely ~~that~~ accidents are as likely to push the outcome to one side or the other, we say ~~that~~ the majority vote outcome is one half of each of the median positions. The average will lie between these median positions while ~~the most~~ the efficiency criterion is satisfied by any position at either of the medians or between them. None of this however affects the conclusions, which hold for any even number above 2.



Majority voting – ½ c ½ d
Efficiency — anywhere between c and d
Average — midway between c and d
Equality — ½ a ½ f
Equal winning — ⅙ a, ⅙ b ··· ⅙ f

Minimum – any of the above

I've taken a set of unequal distribs with equal intensity & unequal intensities with equal distribs (tho' I haven't combined unequal distrib & intensity here — no. of possibilities gets very large).

For each I've worked out ~~most efficient~~

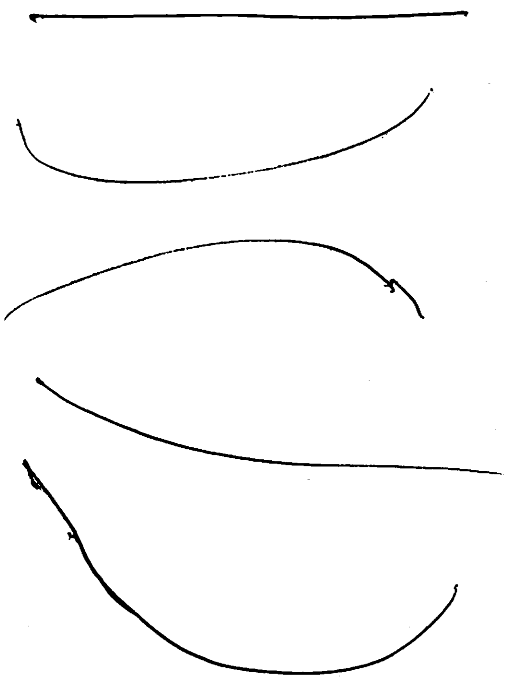
1. Most efficient outcome (ie highest av score)
2. Most equal outcome (ie min deviation from av score)
3. Outcome produced by majority voting
4. Outcome produced by averaging
5. Outcome produced by the procedure giving each person an equal chance of winning

For each of these outcomes I've then calculated

1. Mean score
2. Equality of pay-offs (av deviation from mean)
3. Minimum score received by anybody.

Graph charts 1 against 2, table 1 + 3.

Equal intensity:



(already given)

Anybody can work out the answers for any other distribs he likes fairly quickly but these I thought wd have significant characteristics.

Symmetric polarized

Symmetric consensual

Extreme majority bloc, ~~ext~~ small opposing extreme bloc

Extreme majority bloc, large opposing minority extreme bloc.

Computing positions of outcomes quite easy.

Eg last example

A	B	C	D	E
4	2	1	1	3

MR at median position i.e. at b.

Av : count ~~a~~ a = 1, b = 2 etc

4×1 ~~4~~ 4
2×2 ~~8~~ 4
1×3 ~~11~~ 3
1×4 4
3×5 15

$11 \overline{)30 \cdot 0}$
$2 \cdot 7$ i.e. between B and C

Equality : always for equal intensity $\frac{1}{2}$a $\frac{1}{2}$e

Equal winning : count each outcome proportionally to number of people occupying it

Efficiency : for equal intensity most efficient is always the median i.e. this minimizes deviations.

Given the outcomes we can work out the score of each of the 11 participants. With that we calculate mean score of all the ~~11~~ participants and then when we've got that we work out their average deviation from ~~that~~ mean score. The minimum pay-off can be read off.

Unequal intensity & equal distrib.

Some simplifications

MR always same as av. Outcome of MR always at b. Some other things more difficult – max eff. may not be same outcome as MR, and no simple rule for producing most equal solution.

Example A i.

	A	B	C	D	E	
	1	1	1	1	1	
MR:	4	7	8	7	6	$\frac{32}{5} = 6{\cdot}4$

(Then calculate av dev & see min is 4)

EW Equal satisfaction

	a	×5	b	×2	$\frac{5a+2e}{7}$
A	8	40	0	0	5·7
B	7	35	5	10	6·5
C	6	30	6	12	6·0
D	5	25	7	14	5·7
E	4	20	8	16	5·2
					5·8

(Then calculate av dev., read off min as 5·2)

(1) Reflect on shapes of curves & their implications. What they show is a trade-off between efficiency and equality — the max efficiency and the max equality never coincide. Taking the procedures MR ~~[illegible]~~, AV and EW, MR is always as efficient as anything else and in only one case is there an equally efficient & more equal solution. Average is usually same as MR solution, in the two cases where different ~~always~~ it's between MR and EW; in the case it looks quite good (small loss of efficiency, big gain in equality) but clearly not the general answer to political problems even if it could be used. EW ~~[illegible]~~ with equal intensity always lies between Eff ~~MR~~ and ES — in equal distrib case it's exactly in between (straight line). But with unequal intensities it doesn't ~~make~~ look so good. With Bi, Ci, and Bi Di, it's worse on both ~~[illegible]~~ efficiency and equality than the most equal solution; ~~and~~ in every case (see chart) it gives a lower minimum than the most equal solution; and in every case it gives a minimum no higher than the most efficient solution, and is worse in 3 cases (Bi, Ci, Bi Di)

Points.

~~(1) Main Trade-offs (lines define a Pareto feasibility frontier) — the more efficiency, the less equality and vice versa.~~

(2) Importance of configuration of sentiments in relation to decision processes. With equal distribution A_iE_i the most efficient outcome (combined with an av. dev. of 1.6) is only as efficient (6.0) as the outcome which is totally equal with equal intensity and equal distribution. This is an extreme case among our examples but there are many others ~~where~~ say, ~~equal winning with an configuration [illegible] is a bit more efficient~~ comparisons which show the importance of the configuration. For example, if with equal intensity the distrib is bunched in the middle, you can have EW with the same efficiency and half the av. dev as the most efficient outcome with uniform distribution. And NB these examples are not very extreme cases — we could have more unequal distribs with equal intensity, more

unequal intensities with equal distribution. or we could combine unequal intensity and unequal distribution — something I haven't done at all. Thus, with $\smile$ and E_i, ~~xxx~~ MR ~~efficiency~~ wd be off the picture: & av 5.7, av dev 1.8 ; ~~similarly~~ with $\cup$ and A_iE_i 5.45 and ~~1.78 which is on the picture~~ ~~that~~ 1.6 which is on the picture but very low in both directions.

(3) Tho it's difficult to speak exactly of relative importance, it's clear ~~that~~ the value of outcomes is *relatively* insensitive to the inequality of the distribution compared with the inequality of the intensities. Adding a second person or even two to the end of a five-position continuum has ~~has~~ a quite trivial effect on the values of outcomes, whereas just doubling the intensity of one person at the end of the spectrum has a pretty big effect. In the examples I've contrasted situations with equal distribution and just one or two intense members with much more drastic departures from uniform distribution ~~and res~~ holding intensity equal, and it's clear even so ~~that~~

the uneven distributions cluster around the equal intensity / uniform distribution line much more than do the unequal intensities.

4 Majority voting comes out pretty well under all cirx, tho' better in some than others. This in a way connects up with the first point I made: it's easy to say that majority voting may be acceptable <u>faute de mieux</u> as a workable procedure but that if we could ~~be~~ have complete information about preferences (including their strength) and could ~~manage~~ then apply any rule of aggregation we liked, we'd be able to do much better. What I think my toy societies show pretty conclusively, and I don't see why this isn't generally true, is that given the configuration of sentiments among the population even God couldn't produce an outcome that is better in terms of both equality and efficiency than ordinary voting will produce.

In all our examples there is no ~~ex~~ case where there is a <u>more</u> efficient outcome than that produced by majority voting, and there is only one (A_i) which is equally efficient and better on equality. This is, I think, fairly striking given the range of the examples. (It's not the case ~~that~~ that there couldn't be something more efficient than the outcome of MR — if we stick to A_i but push up A's intensity from ~~2~~ twice to three times we get the result ~~that~~ at ~~E~~ position c (outcome of MR) av score is 6.0, whereas at b it's 6.2 (NB b not a — at a it's back to 6.0). However even this is not very dramatic; where b (and to a less extent a) score here is in equality and minimum. b has a minimum of 5 as against a minimum of 2 with c. This illustrates an important point about MR — that it has a high cost in inequality when there ~~are~~ is one extreme and very intense minority, compared with the degree of equality obtainable. With very intense characters at both extremes there is also a ~~high cost~~ lot of inequality but ~~also~~ the amount of equality obtainable is less. This can be seen by comparing the slopes of ~~a~~ Ai and AiEi.

The strength of MR can be seen more vividly if we turn our attention to minima. With equal intensities ~~whatever~~ and any of the symmetrical ~~the~~ distributions, the minimum with the most efficient ~~so~~ outcome is the same as the minimum with a totally equal outcome. In other words ~~the~~ you can get more equality than in the most equal position but the only way you can get it is simply by bringing everybody down to the position of the worst off – ~~there~~'re no way of actually transferring satisfaction from the best off to the worst off, all you can do is destroy it. With the two non-symmetrical distrdributions and there is a difference between 6 and 5 in the minimum. When we take the unequal intensities we can see ~~that~~ in every case except Ai the minimum with the most efficient solution is the same as ~~that~~ with the most equal; in the case of

Ai it's a difference between 4.0 and 5.2. (With unequal intensity, the most equal is not necessarily the minimax but in fact the highest minimum obtainable is pretty close to 5.2) ~~so we~~

Various other things one could pick out but ~~by~~ these are the ones ~~that~~ seem to me most interesting. I hope that you'll at least have found this analysis suggestive — but just what it suggests is something we might talk about.
